The Quaternion Formulae for Quantification of Curves, Surfaces and Solids, and for Barycentres.

BY W. I. STRINGHAM, *Fellow of the Johns Hopkins University.*

IN order to avoid a clumsy circumlocution, I have ventured to use the word quantification to denote in general that class of operations expressed in the several special cases by the terms, rectification, quadrature and cubature. Some of the quaternion formulae for quantification were given in a paper entitled "Investigations in Quaternions," communicated to the *American Academy of Arts and Sciences*, 9th January, 1878. They are here reproduced in a more general form together with the formulae for barycentres.

Quantification.

Let dM represent in general the element of arc, surface, or volume. Then if $\rho = \psi(t)$ be the vector equation of any curve in space, where t is a scalar variable,

$$dM = ds = \mathrm{T}\rho'.dt, \tag{1}$$

where $\rho' = \mathrm{D}_t\rho =$ the tangent to the curve at the point ρ. The formula for rectification, therefore, is

$$s = \int \mathrm{T}\rho'.dt. \tag{2}$$

The double area of the triangle, two of whose sides are ρ and ρ', is $\mathrm{TV}\rho\rho'$, and the element of area swept by ρ is

$$dM = dA = \frac{1}{2}\,\mathrm{TV}\rho\rho'.dt. \tag{3}$$

Hence for quadrature of plane areas

$$A = \frac{1}{2}\int \mathrm{TV}\rho\rho'.dt. \tag{4}$$

This formula is sufficient to determine the area of a sector whose vertex is at the origin. In order to determine the area of any other sector (the surface being plane) whose vertex is δ, it is only necessary to write $\pi = \rho - \delta$ instead of ρ, δ being the vector of the new origin with reference to the old. Thus, to

determine an area limited by a given chord, make δ the vector of one extremity of the chord and determine the limits of integration by assuming $\rho = \delta$ and $\rho = \delta + \eta$, where η is the chord in question.

A surface in general is represented by the vector equation $\rho = \chi\,(t, u)$, where t, u, are independent scalar variables. If at any instant u remain constant and t vary, ρ will describe a determinate arc upon the surface; if t remain constant and u vary, ρ will describe another arc cutting the former at the point ρ. Hence

$$\rho'_1 = \mathrm{D}_t\rho\,, \quad \rho'_2 = \mathrm{D}_u\rho \tag{5}$$

will represent two tangents to the surface intersecting at their common point of contact. Therefore,

$$dM = dS = \mathrm{TV}\rho'_1\rho'_2\,.\,dtdu \tag{6}$$

will represent an elementary parallelogram upon the surface. Hence, for quadratures in general,

$$S = \int \mathrm{TV}\rho'_1\rho'_2\,.\,dt, u, \tag{7}$$

where dt, u means $dtdu$. The equation $\rho = \chi\,(t, u)$, for the special case of plane surfaces, may be written $\rho = u\psi\,(t) = u\tau$, the origin being on the surface. The formula last written then becomes

$$S = \frac{1}{2}\,[u^2]^1_u \int \mathrm{TV}\tau\tau'.\,dt, \tag{8}$$

where $\tau' = \mathrm{D}_t\tau$. This formula is a more general expression for (4) wherein the limits for u were 0 and 1. It is evident that the choice of any other limits for u would determine an area lying between two similar curves.

If, in the equation of a plane curve, $\rho = \psi\,(t)$, ϕ be the angle of revolution, and the curve be revolved about an axis α, (with the condition $\mathrm{T}\alpha = 1$), then the element of area of the surface of revolution will be

$$dM = dS = \mathrm{TV}\alpha\rho\,.\,d\phi\,.\,\mathrm{T}\rho'.\,dt, \tag{9}$$

whence

$$S = [\phi]^\phi_0 \int \mathrm{T}\rho'\mathrm{V}\alpha\rho\,.\,dt, \tag{10}$$

where $\rho' = \mathrm{D}_t\rho$ as before.

Let now $\omega = z\rho$, where $\rho = \chi\,(t, u)$ and z, t, u are independent scalar variables. When z varies between 0 and 1, the extremity of ω generates the solid enclosed by the surface whose vector generator is ρ. Since $\mathrm{S}\omega'_1\omega'_2\omega'_3$ is the volume of the parallelopiped three of whose edges are ω'_1, ω'_2, ω'_3, then if

$$\mathrm{D}_t\omega = \omega'_1\,, \quad \mathrm{D}_u\omega = \omega'_2\,, \quad \mathrm{D}_z\omega = \omega'_3\,,$$

an elementary parallelopiped within the solid will be represented by

$$dM = dV = \mathrm{S}\omega'_1\omega'_2\omega'_3\,.\,dt, u, z. \tag{11}$$

Hence, for cubatures in general,

$$V = \int\int\int S\omega'_1\omega'_2\omega'_3 . dt, u, z, \tag{12}$$

and since $\omega'_1 = z\rho'_1$, $\omega'_2 = z\rho'_2$, $\omega'_3 = \rho$,

$$\therefore \quad V = \frac{1}{3}\,[z^3]^1_z \int\int S\rho\rho'_1\rho'_2 . dt, u, \tag{13}$$

where $\rho'_1 = D_t\rho$, $\rho'_2 = D_u\rho$. This formula enables us to determine the contents of sectors, or cones whose vertices are at the origin. For the cubature of a portion of the solid limited by a plane, instead of ρ write $\rho - \delta$, where δ is the vector of some point in the given plane, and determine the limits of integration by making ρ satisfy the equation of the plane in question.

The equation, $\rho = u\psi(t)$, of a plane surface, with the condition that the surface be revolved about an axis α, will be sufficient to determine all points of a space enclosed by a surface of revolution. If $T\alpha = 1$ and ϕ be the angle of revolution, $V\alpha\rho$ is the projection of ρ on the normal to the plane of $\alpha\rho$ and $V\alpha\rho . d\phi$ is identical with what $\rho'_2 du$ becomes in this case, that is, it is the tangent to a parallel of latitude. Then

$$dM = dV = S\rho\rho' V\alpha\rho . z^2 dt dz d\phi, \tag{14}$$

$\rho' = D_t\rho$. Hence, for solids of revolution,

$$V = \frac{1}{3}\,[z^3\phi]^1_z \,|^{\phi}_{0} \int S\rho\rho' V\alpha\rho . dt. \tag{15}$$

The order of substitution of limits is $1, z$, afterwards $\phi, 0$. This may also be written

$$V = \frac{1}{3}\,[z^3\phi]^1_z \,|^{\phi}_{0} \int TV\rho\rho' V\alpha\rho . dt; \tag{16}$$

for $S\rho\rho' V\alpha\rho = SV\rho\rho' V\alpha\rho$, and $V(V\rho\rho' V\alpha\rho) = 0$; therefore,

$$S^2 V\rho\rho' V\alpha\rho = T^2 V\rho\rho' V\alpha\rho, \quad \text{or} \quad S\rho\rho' V\alpha\rho = \pm TV\rho\rho' V\alpha\rho.$$

In the formulae for cubatures, the limits 0 and 1 for z give the contents of the space swept by the generating vector of the surface; any other limits would determine the contents of a space lying between two similar surfaces, *i. e.* of a shell.

Barycentres.

In Hamilton's equation*

$$\Sigma m_r (e_r - e) = 0, \quad \text{or} \quad \Sigma m_r \varepsilon_r = 0, \tag{17}$$

wherein the e and e_r are symbols of position in space, the ε_r are vectors to the points e_r from e, and the m_r are scalar quantities, or weights, with which the

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* Elements of Quaternions, p. 89.

points e_r may be regarded as being loaded,—the point e is defined as the barycentre of the system.

Of the system represented by this equation the barycentre is evidently at the origin of vectors. A change of origin, referred to which $\bar{\gamma}$ is the vector to the barycentre, gives

$$\Sigma m_r(\varepsilon_r - \bar{\gamma}) = 0,$$

that is,

$$\bar{\gamma} = \frac{\Sigma m_r \varepsilon_r}{\Sigma m_r}. \tag{18}$$

For a continuous, homogeneous mass, this equation assumes the form

$$\bar{\gamma} = \int \rho . dM \div \int dM, \tag{19}$$

where dM is the element of mass. It remains to substitute in this formula the values already obtained for dM in the several cases above considered.

For arcs $dM = \mathrm{T}\rho'.dt$, (3), and

$$\bar{\gamma} = \int \rho \mathrm{T}\rho'.dt \div \int \mathrm{T}\rho'.dt. \tag{20}$$

For surfaces in general $dM = \mathrm{TV}\rho'_1\rho'_2.dt,u$, (6), and

$$\bar{\gamma} = \int\int \rho \mathrm{TV}\rho'_1\rho'_2.dt,u \div \int\int \mathrm{TV}\rho'_1\rho'_2.dt,u. \tag{21}$$

For plane surfaces (8) this becomes

$$\bar{\gamma} = \frac{1}{3}[u^3]^1_u \int \tau \mathrm{TV}\tau\tau'.dt \div \frac{1}{2}[u^2]^1_u \int \mathrm{TV}\tau\tau'.dt, \tag{22}$$

$u\tau = \rho$, $\tau' = \mathrm{D}_t\tau$. For surfaces of revolution dM is (9), and

$$\bar{\gamma} = \int \rho \mathrm{T}\rho'\mathrm{V}\alpha\rho.dt \div \int \mathrm{T}\rho'\mathrm{V}\alpha\rho.dt, \tag{23}$$

$\alpha =$ axis of revolution, $\mathrm{T}\alpha = 1$. For solids in general dM is (11), and

$$\begin{aligned} \bar{\gamma} &= \int\int\int \omega \mathrm{S}\omega'_1\omega'_2\omega'_3.dt,u,z \div \int\int\int \mathrm{S}\omega'_1\omega'_2\omega'_3.dt,u,z \\ &= \frac{1}{4}[z^4]^1_z \int\int \rho \mathrm{S}\rho\rho'_1\rho'_2.dt,u \div \frac{1}{3}[z^3]^1_z \int\int \mathrm{S}\rho\rho'_1\rho'_2.dt,u, \end{aligned} \tag{24}$$

$\rho'_1 = \mathrm{D}_t\rho$, $\rho'_2 = \mathrm{D}_u\rho$. For solids of revolution dM is (14), and

$$\bar{\gamma} = \frac{1}{4}[z^4]^1_z \int \rho \mathrm{S}\rho\rho'\mathrm{V}\alpha\rho.dt \div \frac{1}{3}[z^3]^1_z \int \mathrm{S}\rho\rho'\mathrm{V}\alpha\rho.dt, \tag{25}$$

$\rho' = \mathrm{D}_t\rho$. This last formula [see (16)] may also be written

$$\bar{\gamma} = \frac{1}{4}[z^4]^1_z \int \rho \mathrm{TV}\rho\rho'\mathrm{V}\alpha\rho.dt \div \frac{1}{3}[z^3]^1_z \int \mathrm{TV}\rho\rho'\mathrm{V}\alpha\rho.dt. \tag{26}$$

It is to be noticed that the formulae (24), (25), (26) give the barycentres of the entire solids, or of shells, according as the limits assumed for z are 0

and 1, or some other. Suppose the limits to be z and $z+\varepsilon$, where ε is infinitesimal. If the terms containing the second and higher powers of ε be neglected, there will be in the numerator the factor $(z+\varepsilon)^4 - z^4 = 4z^3\varepsilon$, and in the denominator, $(z+\varepsilon)^3 - z^3 = 3z^2\varepsilon$; and (24) will reduce to

$$\bar{\gamma} = z\int\int \rho S\rho\rho'_1\rho'_2 . dt, u \div \int\int S\rho\rho'_1\rho'_2 . dt, u, \tag{27}$$

the barycentric vector to an infinitely thin shell. The relative thickness of the shell, as shown by the factor z in the equation, $\omega = z\chi\,(t,u)$, of the shell, will be determined by the position of the origin of the generating vector.

Applications.

The ellipsoid will afford a convenient illustration of the application of the above methods. Its equation is

$$\rho = \alpha\cos x + \sigma\sin x\,,$$

where

$$\sigma = \beta\cos y + \gamma\sin y\,,$$

and α, β, γ are the three principal semiaxes. By differentiation and reduction

$$TV\rho'_1\rho'_2 = as\sin x\sqrt{1-\sin^2 w\cos^2 x}\,,$$

where $a = T\alpha$, $b = T\beta$, $c = T\gamma$, $s = T\sigma'$, and $\sin^2 w = 1 - \frac{b^2c^2}{a^2s^2}$, or $\cos w = \frac{bc}{as}$. Hence by (7)

$$S = \int \frac{bc}{\cos w}\int \sin x\sqrt{1-\sin^2 w\cos^2 x}\,.\,dx, y\,.$$

Let $\sin v = \sin w\cos x$, $\cos v = \sqrt{1-\sin^2 w\cos^2 x}$. Then

$$S = \int \frac{2bc}{\sin 2w}\int \cos^2 v\,.\,dv, y = \int \frac{bc}{2}\,\frac{[2v+\sin 2v]_0^v}{\sin 2w}\,.\,dy\,. \tag{28}$$

When the surface is one of revolution, say the prolate ellipsoid, then $b=c=s$, $\sin^2 w = \frac{a^2-b^2}{a^2}$, and the expression (28) reduces to

$$S = \frac{a^2b}{4\sqrt{a^2-b^2}}\,[y\,(2v+\sin 2v)]_0^v\big|_0^y\,.$$

The order of substitution of limits is v, 0, afterwards y, 0.

By easy transformations it will be found that $V\rho\rho'_1 = V\alpha\sigma$, $V\sigma\sigma' = V\beta\gamma$, and therefore

$$S\rho\rho'_1\rho'_2 = abc\sin x\,.$$

Hence, for the ellipsoidal shell [see (13)],

$$V = \frac{abc}{3}\,[z^3]_z^1\int\int \sin x\,.\,dx, y = \frac{abc}{3}\,[z^3y\cos x]_0^x\big|_0^y\big|_z^1\,.$$

The order of substitution of limits is x, 0; y, 0; 1, z.

The value of $\rho S\rho\rho'_1\rho'_2$ is abc $(\alpha \sin x \cos x + \sigma \sin^2 x)$, and its complete integral is

$$\frac{abc}{4}\left[2\alpha y \sin^2 x + (\beta \sin y - \gamma \cos y)(2x - \sin 2x)\right]_0^x\Big|_0^y .$$

Hence the barycentric vector of the ellipsoidal shell is

$$\bar{\gamma} = \frac{3\left[z^4\{2\alpha y \sin^2 x + (\beta \sin y - \gamma \cos y)(2x - \sin 2x)\}\right]_0^x\Big|_0^y\Big|_z^1}{16\left[z^3 y \cos x\right]_0^x\Big|_0^y\Big|_z^1} .$$

The order of substitution of limits is important; it is as above in the expression for V. If the shell be infinitely thin, in this case an ellipsoidal Chaslesian shell, the application of formula (27) gives

$$\bar{\gamma} = \frac{z\left[2\alpha y \sin^2 x + (\beta \sin y - \gamma \cos y)(2x - \sin 2x)\right]_0^x\Big|_0^y}{4\left[y \cos x\right]_0^x\Big|_0^y} .$$

For the ellipsoidal solid $\bar{\gamma}$ is the expression last written multiplied by $\frac{3}{4z}$. The barycentric vector to the half solid, bounded by the plane of $\alpha\beta$, will easily be found to have the value $\frac{3}{8}\gamma$, and that of one-eighth of the solid, bounded by the planes $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$, to have the value $\frac{3}{8}(\alpha+\beta+\gamma)$.

It is worth while to observe, in this place, that from the equation $\rho = q - w$, (where ρ is the generating vector to a family of surfaces, and w is a scalar parameter), which Tait* has assigned as the equation of a *volume*, would be deduced substantially the same results as the foregoing. Let Tq represent the length of the radius vector to the surface which bounds the solid, and let q be written as a function of t, u, θ; then if t, u be taken as the variables which define the surface, Tq and UVq will be functions of t, u exclusively, and θ will turn out to be the angle of q. Thus the equation of the solid may be written

$$q = \phi(t, u, \theta),$$

or

$$Vq = \chi(t, u) \sin\theta .$$

This vector equation will be satisfied for every point within the solid; and it has the same form as the equation $\rho = z\chi(t, u)$, made use of in the foregoing discussion.

*Treatise on Quaternions, p. 61.

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